Approximate methods

In exact method difficulties arises in

- Solving roots of the characteristic equation. Except for very simple boundary conditions, one has to go for numerical solution.
- In determining the normal modes of the system
- Determination of steady state response

So for quick determination of the natural frequencies of a system, when a very accurate result is not of much importance one should go for an approximate method.

Approximation → Modeling

Approximation → Series solution

Approximate method where approximation error should be within acceptable limits one may assume a series solution as

$$u(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x) \qquad (1)$$

where $\phi_n(x)$ is the normal modes and $f_n(t)$ is the time function which depends upon initial conditions and forcing function. There are certain difficulties that limit the application of classical analysis of continua to a very simple geometry only.

- The infinite series sometimes converge very slowly and it is difficult to estimate how many terms are needed for engineering accuracy.
- The formulation and computation efforts are prohibitive for systems of engineering complexity.

The special methods (as approximate methods) treat the continuous systems, for vibration analysis purpose, as discrete systems. This can be done with one of the following methods.

- Taking only *n* natural modes and considering them as generalized coordinates and then computing the *n* weighing functions $f_n(t)$ to best fit the initial conditions or the forcing functions.
- Considering *n* known functions $\phi_n(x)$ that satisfy the geometric boundary conditions of the system and then computing the functions $f_n(t)$ to best fit the

differential equation, the remaining boundary conditions, and the initial conditions of the forcing functions.

- Taking the physical coordinates of n number of points of the system q, ($q_1, \cdots q_n$) as the generalized coordinates, considering them as functions of time, and computing them to fit the differential equation; and the initial and boundary conditions.

- The main advantage of all theses methods is that instead of dealing with one or more partial differential equations, one deals with a larger number of ordinary differential equations usually liner with constant coefficients, which are particularly suitable for solution in fast computing machines.

Rayleigh's Method

-Rayleigh method gives a fast and rather accurate computation of the fundamental frequency of the system.

-It applies for both discrete and continuous systems.

Consider a discrete, conservative system described by the matrix equation

$$Mx + Kx = 0 \qquad (2)$$

The equation above is satisfied by a set of n eigenvalues ω_i^2 and normalized eigenvectors ϕ^i, which satisfy the equation

$$K\phi_i = \omega_2^2 M\phi_i \qquad i = 1, 2\text{----}n \qquad (3)$$

Multiplying both sides of (3) by ϕ_i' and dividing by a scalar $\phi_i' M \phi_i$, which is a quadratic form, we have

$$\omega_i^2 = \frac{\phi_i' K \phi_i}{\phi_i' M \phi_i} \qquad (4)$$

If we know the eigenvector ϕ^i, we can obtain the corresponding eigenvalue ω_i^2 by eq^n(4).

However in general, the eigenvectors are not known and one has to find it for the particular system. Suppose that we consider an arbitrary vector Z in eq^n (4). So eqn(4) can be written as

$$\omega^2 = R(z) = \frac{z'Kz}{z'Mz}. \qquad (5)$$

Here *R(z)* depends on the vector z and is called Rayleigh's quotient . When the vector z coincides with an eigenvector ϕ_i , Rayleigh's quotient coincides with the corresponding eigenvalues.

From vector algebra it is known that vector z can be expressed as a linear combination of independent vectors.

$$z = c_1 z_1 + c_2 z_2 + \cdots + c_n z_n = \sum_{i=1}^{n} c_i z_i = Zc \tag{6}$$

where Z is a square modal matrix $[z_1 z_2 \cdots z_n]$ and $c = [c_1 c_2 \cdots c_n]$. If the vector z_i have been normalized so that

$$Z^T MZ = I, \text{ then } Z^T KZ = \text{diag } [\omega_1^2, \omega_2^2 ....\omega_n^2] = P \tag{7}$$

using (7) in (6) and using orthogonal Property

$$R(z) = \frac{c'Z'KZc}{c'Z'MZc} = \frac{c'Pc}{c'Ic} = \frac{\sum_{i=1}^{n} c_i \omega_i^2}{\sum_{i=1}^{n} c_i^{\,2}} \tag{8}$$

Equation (6) is similar to the free vibration response of a system which contains all the normal modes. In the assumed function let the rth mode deviates from the actual mode. So the c_r will be larger in comparison to the other c_i's. Now taking $\frac{c_i}{c_r} = \varepsilon_i, \quad \varepsilon_i$ □ 1, equation (8) can be written as

$$R_r(z) = \frac{\omega_r^2 + \sum_{i=1}^{n} \left(\frac{c_i}{c_r} \right)^2 \omega_i^2}{\sum_{i=1}^{n} \left(\frac{c_i}{c_r} \right)^2} = \frac{\omega_r^2 + \sum_{i=1}^{n} \varepsilon_i^{\,2} \omega_i^2}{\sum_{i=1}^{n} \varepsilon_i^{\,2}} \quad i \neq r \tag{9}$$

For $r = 1$,

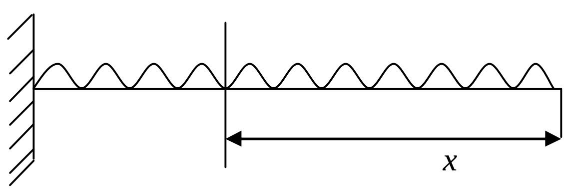
$$R_1(z) = \omega_1^2 \frac{1 + \sum_{i=2}^{n} \varepsilon_i^{\,2} \omega_i^2}{1 + \sum_{i=1}^{n} \varepsilon_i^{\,2}} \tag{10}$$

From equation (9) it is apparent that the Rayleigh quotient differs from the eigenvalues ω_r^2 by second order terms of the error of the eigenvector. In other words, if the error in selecting the nonexact eigenvector is ε, then the error in the eigenvalues is ε. So a 20% error in the form of eigen vector will yield a 4% error in the eigenvalues and a corresponding 10 % error in eigen value will result in 1% error in eigenvector. From eqn(10), it may also be concluded that

$$R_1(z) \geq \omega_1^2 \qquad (11)$$

Hence the Rayleigh's quotient is never lower than the smallest eigenvalues. Hence this method gives the upper bound approximation of the fundamental frequency of the system.
Rayleigh's principle can be stated as in a conservative system the frequency of vibration has a stationary value in neighbourhood of a natural mode.

Example: Using Rayleigh quotient method, find the fundamental frequency for a cantilever beam assuming the approximate function as the static deflection curve.



Static deflection of a cantilever beam can be found using bending equation as follows.

$$M = wx.\frac{x}{2} = EI\frac{d^2y}{dx^2}$$ where w is the load per unit length.

or, $$EI\frac{dy}{dx} = \frac{1}{2}\frac{wx^3}{3} + C_1$$

$$EIy = \frac{1}{6}\frac{wx^4}{4} + C_1x + C_2$$

BCs $x = l$ $$\left.\begin{matrix} y = 0 \\ \dfrac{dy}{dx} = 0 \end{matrix}\right\} \Rightarrow \qquad C_1 = \frac{-1}{6}wl^3$$

$$C_2 = -\left(\frac{1}{6_\times}\ \frac{wx^4}{4} + C_1 x\right)\Big| x = l$$

$$= -\left(\frac{wl^4}{24} - \frac{\perp}{6} wl^4\right) = +\frac{wl^4}{8}$$

$$\Rightarrow y = \frac{w}{24EI}\left(x^4 - 4l^3 x + 3l^4\right)$$ (Deflection from free end)

To measure x from fixed end

One may substitute $x = (l - x')$ in the above equation.

$$\therefore y = \frac{w}{24ET}\left[(l - x')^4 - 4l^3(l - x') + 3l^4\right]$$

$$= \frac{w}{24EI}\left[\left(l^3 - x^3 - 3l^2 x + 3lx^2 - 4l^3\right)\right]$$ replacing x’ by x

$$= \frac{w}{24EI}\left[-3l^4 - x^3 l - 3l^3 x + 3l^2 x^2 + 3l^3 x + x^4 + 3l^2 x^2 - 3lx^3 + 3l^4\right]$$

$$= \frac{w}{24EI}\left[x^4 - 4lx^3 + 6l^2 x^2\right]$$

Taking static deflection curve as $\varphi(x) = \frac{w}{24EI}\left(x^4 - 4x^3 l + 6l^2 x^2\right)$

$$\frac{d\varphi(x)}{dx} = \frac{w}{24EI}(4x^3 - 12lx^2 + 12l^2 x)$$ $w =$ weight/unit length

$$\frac{d^2\varphi(x)}{dx^2} = \frac{w}{24EI}(12x^2 - 24lx + 12l^2) \qquad = \frac{w}{2EI}(x^2 - 2lx + l^2)$$

Potential energy $= \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$

Kinetic energy $T = \frac{1}{2}\omega^2 \int_o^l m(x) \quad y^2 dx$

$$\Rightarrow \omega^2 = \frac{\frac{1}{2}\int_0^l EI . \frac{w}{2EI}(x - l)^4 dx}{\frac{1}{2}\int_o^l m(x^4 - 4lx^3 + 6l^2 x^2)^2 \frac{w^2 dx}{24^2 E^2 I^2}} = \frac{I_1}{I_2}$$

$$I_1 = -\frac{1}{2} - \int_l^o \frac{EIw}{2EI}(t)^4 \, dt \qquad \begin{array}{l} t - x = t \rightarrow dt = -dx \\ x = 0, \quad t = +l \\ x = l \quad t = o \end{array}$$

$$= -\frac{1}{2}\left[E\frac{w}{2}\frac{t^5}{5}\right]_l^o = -\frac{1}{2}\frac{w}{2\times 10}\left(o - l^5\right) = \frac{wl^5}{20}$$

$$I_2 = \frac{1}{2}\frac{mw^2}{(24EI)^2}\int_0^l (x^8 + 16l^2x^6 + 36l^4x^4 - 2\times 4lx^7 - 48l^2x^5 + 12lx^6)dx$$

$$= \frac{1}{2}m\left[\frac{x^9}{9} + \frac{16}{7}l^2x^7 + \frac{36}{5}l^4x^5 - \frac{8lx^8}{8} - \frac{{}^8 48l^3x^6}{6} + \frac{12l^2x^7}{7}\right]_o^l \frac{w^2}{(24EI)^2}$$

$$= \frac{1}{2}m\left(\frac{l^9}{9} + \frac{16l^9}{7} + \frac{36l^9}{5} - \frac{8l^9}{8} - 8l^9 + \frac{12}{7}l^9\right)\frac{w^2}{(24EI)^2}$$

$$= \frac{1}{2}m^{l9}\left(\frac{1}{9} + \frac{16}{17} + \frac{36}{5} - 1 - 8 + \frac{12}{7}\right)\left(\frac{w^2}{(24EI)^2}\right)$$

$$= \frac{0.96657}{2} \times \frac{ml^9 \times w^2}{(24EI)^2}$$

$$\Rightarrow \omega^2 = \frac{I_1}{I_2} = \frac{\omega l^5 \times 2 \times (24EI)^2}{20 \times mw^2 \times 0.96657t^9}$$

$$\omega^2 = \frac{59.591963(EI)^2}{ml^5} \quad \text{Ans}$$

Rayleigh's upper bound approximation for Lumped mass system

Rayleigh method can be used to determine the fundamental frequency of a beam or shaft represented by a series of lumped masses. Let $y_{1,}\, y_2 \cdots y_n$ are the maximum static deflection under the concentrated load $m_1g, m_2g \cdots m_ng$ as shown in figure 2.Here, the mass of the beam is neglected.

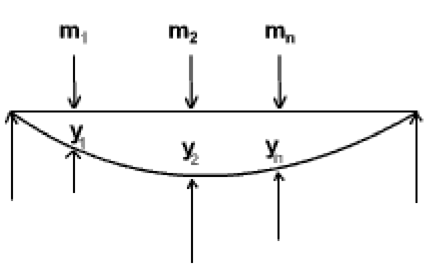


Fig2

The max potential energy $= \frac{g}{2}(m_1 y_1 + m_1 y_1 + ...) = \frac{g}{2}\sum my$

Max Kinetic Energy. $= \frac{1}{2}m_1 v_1^{\,2} + \frac{1}{2}m_2 v_2^{\,2} + \frac{1}{2}m_3 v_3^{\,2} + ...$

$$= \frac{1}{2}m_1(\omega y_1)^2 + \frac{1}{2}m_2(\omega y_2)^2 + \frac{1}{2}m_3(\omega y_3)^2 + ...$$

$$= \frac{1}{2}\omega^2 \sum my^2$$

Now equating the maximum kinetic energy to the maximum potential energy

$$=> \boxed{\frac{g}{2}\sum my = \frac{1}{2}\omega^2 \sum my^2}$$

$$=> \boxed{\omega_n^{\,2} = \frac{g\sum my}{\sum my^2}}$$

Unlike Dunkerley's formula, which is valid for lateral vibration of shafts only, Rayleigh's method is valid for a system performing oscillatory motion in any manner i.e., bending, torsional or longitudinal motions.

Example2: Find the fundamental frequency of the simply supported beam of length l carrying three discs of mass $m, 2m$ and m equidistantly placed from the left end.

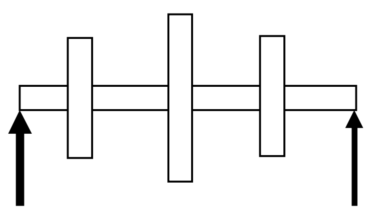


Figure 3.

Solution

Consider the shaft carrying three discs as shown in the figure. The influence coefficients are,

$$a_{11} = \frac{3l^3}{256EI}, \quad a_{22} = \frac{l^3}{48EI}, \quad a_{33} = \frac{3l^3}{256EI}$$

Influence Coefficients:
a_{ij} deflection at station i due to unit load at station j

Using Dunkerley's formula

$$\frac{1}{\omega_1^2}=\frac{3ml^3}{256EI},\ \frac{1}{\omega_2^2}=\frac{2ml^3}{48EI},\ \frac{1}{\omega_3^2}=\frac{3ml^3}{256EI}$$

$$\frac{1}{\omega_n^2}=\frac{3ml^3}{256EI}+\frac{2ml^3}{48EI}+\frac{3ml^3}{256EI}$$

$$=\frac{(3+10.66+3)\text{ml}^3}{256\text{EI}}$$

$$\omega_n^2=\frac{256}{16.66}\frac{EI}{ml^3}=15.36\frac{EI}{ml^3}$$

$$\boxed{\omega_n=3.9191\sqrt{\frac{EI}{ml^3}}}$$

$$\omega_n^2=15.36\frac{EI}{ml^3}\ \text{Dunkerly}$$
$$=\frac{16.199\text{EI}}{ml^3}\ \text{Exact}$$

Rayleigh method:

Flexibility influence coefficient displacement at i due to unit load at j with all other forces equal to zero.

$$a_{11}=\frac{9l^3}{768EI},\ a_{12}=\frac{11l^3}{768EI},\ a_{13}=\frac{7l^3}{768EI}$$

$$a_{22}=\frac{16l^3}{768EI},\ a_{23}=\frac{11l^3}{768EI},\ a_{33}=\frac{9l^3}{768EI}$$

By Maxwell's reciprocal theorem the remaining influence coefficients can easily be determined.

The static deflections are therefore given by,

$$X_1=m_1ga_{11}+m_2ga_{12}+m_3ga_{13}$$

$$X_2=m_1ga_{21}+m_2ga_{22}+m_3ga_{23}$$

$$X_3=m_1ga_{31}+m_2ga_{32}+m_3ga_{33}$$

$m_1=m_3=m$ and $m_2=2m$

$$X_1=\frac{38ml^3g}{768EI},\ X_2=\frac{54ml^3g}{768EI},\ X_3=\frac{38ml^3g}{768EI}$$

$$=> \omega_n^{\ 2} = \frac{g\sum_{i=1}^{n} m_i X_i}{\sum_{i=1}^{n} m_i X_i} = \frac{16.2055\text{EI}}{ml^3}$$

(which is slightly higher than the exact value $16.199\frac{\text{EI}}{\text{ml}^3}$)

Example:

A steam turbine blade of length l, can be considered as a uniform cantilever beam, mass *m* per unit length with a tip mass *M*. The flexural rigidity of the blades is EI. Determine fundamental bending frequency.(Use Rayleigh Method)

Sol:

$$\omega_n = \sqrt{\frac{g}{\Delta}}$$

$$=> \omega_n^{\ 2} = \frac{g}{\Delta} = \frac{g.256EI}{3mgl^3} = \frac{256EI}{3mgl^3}$$

Assuming Y(x,t)=Y(x)cosωt

$$\text{Y(x)=A(1-cos}\frac{\pi \text{x}}{2\text{l}})$$

$$\dot{\text{Y}}(x,t) = -\omega \text{Y(x)sin}\omega\text{t=-}\omega\text{A(1-cos}\frac{\pi\text{x}}{2\text{l}})\sin\omega t$$

$$K.E. = T = \int_0^l \frac{1}{2} m \dot{y}^2\, dx$$

$$= \int_0^l \frac{1}{2} m\omega^2 (1-\cos\frac{\pi x}{2l})^2 A^2 dx$$

$$= \frac{mA^2\omega^2}{2}\int_0^l (1-\cos\frac{\pi x}{2l})^2 dx$$

$$\frac{mA^2\omega^2}{2}\left[x\Big|_0^{\,l} - \left(\frac{2\sin\left(\frac{\pi x}{2l}\right)}{\frac{\pi}{2l}}\right)\Bigg|_0^{\,l} + \frac{x}{2}\Big|_0^{\,l} - \frac{\sin 2(\frac{\pi x}{2l})}{2(\frac{\pi}{2})}\Bigg|_0^{\,l}\right]$$

$$= \frac{mA^2\omega^2}{2}\left[\left(\frac{3}{2}\right)l - \frac{2l}{\pi}.2(1-0) - \frac{l}{\pi}(0-0)\right]$$

$$= \frac{mA^2\omega^2}{2}\left(\frac{3}{2}l - \frac{4l}{\pi}\right) = mA^2\omega^2 l(\frac{3}{4} - \frac{2}{\pi})$$

The K.E. of tipmass $= \frac{1}{2}M(\dot{y}(l)) = \frac{1}{2}M\{\omega A\}^2 = \frac{1}{2}M\omega^2 A^2$

The K.E. of the system $= mA^2\omega^2 l\left(\frac{3}{4}-\frac{2}{\pi}\right)+\frac{1}{2}M\omega^2 A^2$

P.E. $V=\frac{1}{2}EI\int_0^l\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{\pi^4}{64}\frac{EI}{l^3}A^2$

Strain Energy

Equating max P.E. with max K.E.,

$$\omega^2 = \frac{3.0382EI}{(M+0.232ml)l^3}$$

Lecture 2

Dunkerley’s Method (Semi empirical) approximate solution

Let $W_1, W_2, \ldots\ldots W_n$ be the concentrated loads on the shaft due to masses $m_1, m_2, \ldots\ m_n$ and $\Delta_1, \Delta_2, \ldots\ldots\ \Delta_3$ are the static deflections of the shaft under each load. Also let the shaft carry a uniformly distributed mass of m per unit length over its whole span and static deflection at the mid span due to the load of this mass be Δs. Also

Let

ω_n = Frequency of transverse vibration of the whole system.

ω_{ns} = Frequency with distributed load acting alone

ω_{n1}, ω_{n2}...... = Frequency of transverse vibration when each of $W_1, W_2, W_3 \ldots$ act alone.

According to Dunkerley's empirical formula

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_{n_1}^2} + \frac{1}{\omega_{n_2}^2} + \ldots\ldots\ldots + \frac{1}{\omega_{n_s}^2}$$

$$\frac{1}{f_n^2} = \frac{1}{f_{n_1}^2} + \frac{1}{f_{n_2}^2} + \ldots\ldots\ldots + \frac{1}{f_{n_s}^2}$$

Dunkerley's method gives lower bound approximation.

For a simply supported Euler Bernoulli's beam

$$\omega_n^2 = (n\pi)^2 \sqrt{\frac{EI}{\rho L^4}}$$

$$=> \boxed{\omega_n = n\pi \sqrt[4]{\frac{EI}{\rho L^4}}}$$

for simply supported beam with uniformly distributed load, maximum deflection occur at midpoint.

$$\Delta = \frac{5WL^3}{384EI} \quad \text{W = total weight}$$

$$= 5\rho g L^4 / 384EI$$

So, $\dfrac{EI}{\rho L^4} = \dfrac{5g}{384\Delta}$

Hence, $\omega_n^2 = n^2\pi^2\sqrt{\frac{5g}{384\Delta}}$

Similarly for a fixed-fixed beam with loading the maximum deflection can be given by

$$\Delta = \frac{\rho g l^4}{384EI} \Rightarrow \frac{EI}{\rho l^4} = \frac{g}{384\Delta}$$

In this case for the first mode $\omega_n^2 = 22.4\sqrt{\frac{EI}{\rho L^4}}$

So, $\omega_n^2 = 22.4\sqrt{\frac{g}{384\Delta}} = 1.143\sqrt{\frac{g}{384\Delta}}$

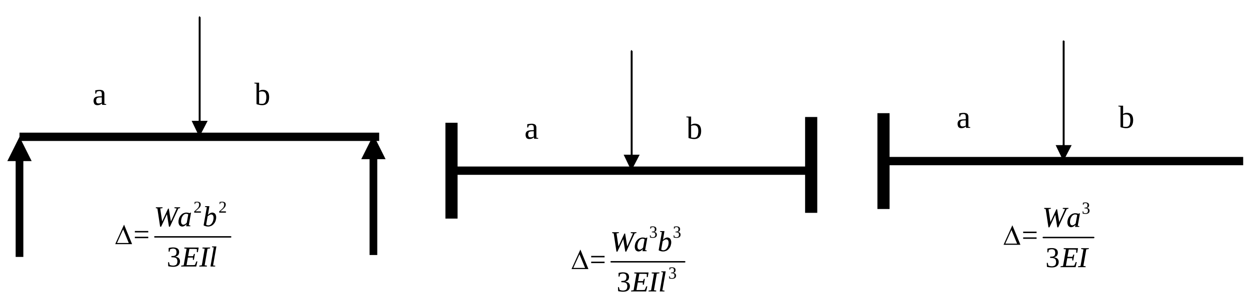
For Cantilever Beam

$$\Delta = \frac{\rho g l^4}{8EI} \Rightarrow \frac{EI}{\rho l^4} = \frac{g}{8\Delta}$$

In this case for the first mode $\omega_n^2 = 3.52\sqrt{\frac{EI}{\rho L^4}}$

So, $\omega_n^2 = 3.52\sqrt{\frac{g}{8\Delta}} = 1.2445\sqrt{\frac{g}{\Delta}}$

In case of concentrated loading the natural frequencies can be determined from the relation $\omega_n^2 = \sqrt{\frac{g}{\Delta}}$, where Δ is the deflection under that load. One may note for the commonly used cases.



Example2: Find the fundamental frequency of the simply supported beam of length l carrying three discs of mass $m, 2m$ and m equidistantly placed from the left end.

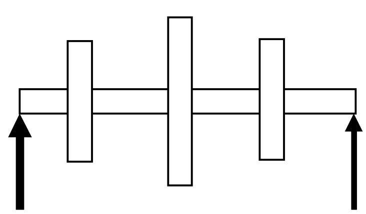


Figure 3.

Solution

Consider the shaft carrying three discs as shown in the figure. The influence coefficients are,

$$a_{11} = \frac{3l^3}{256EI}, \quad a_{22} = \frac{l^3}{48EI}, \quad a_{33} = \frac{3l^3}{256EI}$$

Influence Coefficients:
a_{ij} deflection at station i due to unit load at station j

Using Dunkerley's formula

$$\frac{1}{\omega_1^2} = \frac{3ml^3}{256EI}, \quad \frac{1}{\omega_2^2} = \frac{2ml^3}{48EI}, \quad \frac{1}{\omega_3^2} = \frac{3ml^3}{256EI}$$

$$\frac{1}{\omega_n^2} = \frac{3ml^3}{256EI} + \frac{2ml^3}{48EI} + \frac{3ml^3}{256EI}$$

$$= \frac{(3+10.66+3)\text{ml}^3}{256\text{EI}}$$

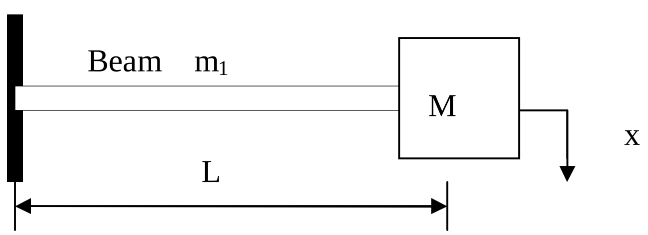
$$\omega_n^2 = \frac{256}{16.66}\frac{EI}{ml^3} = 15.36\frac{EI}{ml^3}$$

$$\boxed{\omega_n = 3.9191\sqrt{\frac{EI}{ml^3}}}$$

$$\omega_n^2 = 15.36\frac{EI}{ml^3} \quad \text{Dunkerly}$$

$$= \frac{16.199\text{EI}}{ml^3} \quad \text{Exact}$$

Example 3:



The natural frequency of a cantilever beam of negligible mass with a concentrated mass M attached is

$$\left(\frac{1}{\omega_{11}^2}\right) = \frac{3EI}{mL^3}$$

And
the natural frequency of a cantilever beam of mass m_1 is

$$\left(\frac{1}{\omega_{22}^{2}}\right)=12.7\frac{EI}{m_1L^3}$$

Therefore, fundamental frequency of the system, is

$$\left(\frac{1}{\omega_{1}^{2}}\right)=\frac{3EI}{mL^3}+12.7\frac{EI}{m_1L^3}=0.412\frac{mL^3}{EI}$$

3. **The Rayleigh-Ritz method**

This is considered as an extension of Rayleigh's method. A closer approximation to the natural mode can be obtained by superposing a number of assumed functions than using by a single assume functions as in Rayleigh's method.

It gives the more accurate result than the previous method.

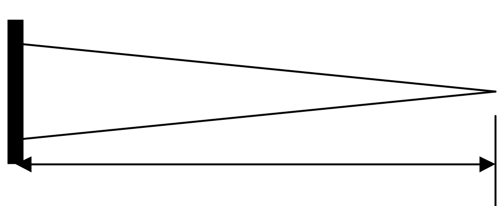
In the case of transverse vibration of beams, if n functions are chosen for approximating the deflection $W(x)$, can be written as

$$w(x)=c_1w_1(x)+c_2w_2(x)+\cdots+c_nw_n(x)$$

Where, $w_1(x), w_2(x), \cdots w_n(x)$ are linear independent functions of the spatial coordinate x which satisfy the boundary condition of the problem, and $c_1, c_2 \cdots, c_n$ are the coefficient to be found.

As the Rayleigh quotients have stationary value near the natural mode by differentient by differentiating the Rayleigh quotient with respect to these coefficients will yield a set of homogeneous algebraic equations, which can be solved to obtain the frequencies.

Example: Find the modal frequencies of a tapered cantilever beam of maximum height h, length l with unity width.



Solution l

Tapered cantilever beam
maximum height h, width =unity., length l

Area of cross section $A(x)=\frac{hx}{l}$

Moment of inertia at any section $I(x)=\frac{1}{12}\left(\frac{hx}{l}\right)^3$

Assuming the deflection function $w_1(x), w_2(x)$ as

$$w_1(x)=\left(1-\frac{x}{l}\right)^2 \text{---------1(a)}$$

$$w_2(x)=\frac{x}{l}\left(1-\frac{x}{l}\right)^2 \text{------------------1(b)}$$

By using one term approximation, same result comes out as in case o f Rayleigh's method

We here using two term approximation,

$$w(x)=c_1\left(1-\frac{x}{l}\right)^2+c_2\frac{x}{l}\left(1-\frac{x}{l}\right)^2 \qquad \text{-----------1(d)}$$

Reyleigh quotient is given by

$$R=\omega^2=\frac{\int_0^l EI(x)\left(\frac{d^2w(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(x)\left(w(x)\right)^2 dx}=\frac{X}{Y} \text{--------------1(e)}$$

Substituting equation 1(d) into the equation 1(e), we obtain

$$X=\frac{Eh^3}{3l^3}\left(\frac{c_1^{\ 2}}{4}+\frac{c_2^{\ 2}}{10}+\frac{c_1c_2}{5}\right) \text{ And } Y=\rho hl\left(\frac{c_1^{\ 2}}{30}+\frac{c_2^{\ 2}}{280}+\frac{2c_1c_2}{105}\right) \text{----1(,f,g)}$$

The condition that makes ω^2 stationary are

$$\frac{\partial(\omega^2)}{\partial c_1}=\frac{Y\frac{\partial X}{\partial c_1}-X\frac{\partial Y}{\partial c_1}}{Y^2}=0 \text{ ------------1(h)}$$

and

$$\frac{\partial(\omega^2)}{\partial c_2} = \frac{Y\dfrac{\partial X}{\partial c_2} - X\dfrac{\partial Y}{\partial c_2}}{Y^2} = 0 \text{--------------1(i)}$$

Substituting the equations 1(f,g) into the equations 1(h,i), can be written as

$$\begin{bmatrix} \left(\frac{1}{2}-\frac{1}{15}\lambda\right) & \left(\frac{1}{5}-\frac{2}{105}\lambda\right) \\ \left(\frac{1}{5}-\frac{2}{105}\lambda\right) & \left(\frac{1}{5}-\frac{1}{140}\lambda\right) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{--------------1(j)}$$

Where $\lambda = \dfrac{3\omega^2 \rho l^4}{Eh^2}$

By setting the **determinant of matrix** in 1(j) equal to zero

We have

$$\frac{1}{8820}\lambda^2 - \frac{13}{1400}\lambda + \frac{3}{50} = 0$$

Therefore, the **natural frequencie**s of the tapered beam are

$$\omega_1 \cong 1.537\left(\frac{Eh^2}{\rho l^4}\right)^{1/2} \text{ and } \omega_2 \cong 4.994\left(\frac{Eh^2}{\rho l^4}\right)^{1/2}$$

Exercise Problem : Find the first three mode frequencies of the tapered beamby using Rayleigh-Ritz method considering the following approximate function.

$$w_1(x) = \left(1-\frac{x}{l}\right)^2 \text{---------1(a)}$$

$$w_2(x) = \frac{x}{l}\left(1-\frac{x}{l}\right)^2 \text{------------------1(b)}$$

$$w_3(x) = \left(\frac{x}{l}\right)^2\left(1-\frac{x}{l}\right)^2 \text{--}$$

Lecture 3

4. Galerkin’s method

In Galerkin’s method the residue obtained by using the assumed mode in the governing differential equation is minimized. Let the assumed shape function of the system be written as

$$\varphi(x) = \sum_{i=1}^{n} c_i \varphi_i(x) \text{ (a)}$$

where $\varphi_i(x)$ is the approximate solution of the differential equation. For example considering the lateral vibration of a beam the differential equation of motion can be written as

$$L(x) = \frac{d^4\phi(x)}{dx^4} - \frac{m\omega^2}{EI}\phi(x) = 0 \text{ .(b)}$$

Similarly for torsional vibration of rod, longitudinal vibration of rod and lateral vibration of taut string one use the following equation.

$$L(x) = \frac{d^2\phi(x)}{dx^2} + \left(\frac{\omega}{c}\right)^2 \phi(x) = 0 \quad \text{(c)}$$

Here $\phi(x)$ is the eigenfunction of the system. When an approximate function $\varphi_i(x)$ is taken, then it will not satisfy the above equation.

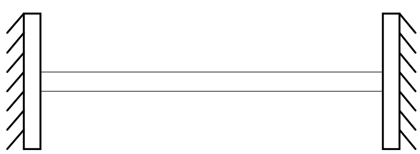
For each function, making the residual (R_i) equal to zero one may obtain the frequencies.

$$R_i = \int_0^l c_i \varphi_i(x) dx = 0 \qquad \text{(d)}$$

Now we have *n* linear and homogeneous equations coefficients $c_1, c_2 \cdots, c_n$. The following example illustrates the application of *Galerkin's* method.

Example

Determine the natural frequencies of a fixed-fixed beam using Galerkin's method.



Taking a function $\varphi_1(x) = x^2(l-x)^2$ which satisfy both the boundary conditions,

$$L(x) = \frac{d^4\phi(x)}{dx^4} - \frac{m\omega^2}{EI}\phi(x) = \frac{d^4\phi(x)}{dx^4} - \beta^4\phi(x)$$

$$L(x) = \beta^4\left(-x^4l^4 - 6x^6l^2 + 4x^5l^3 - x^8 + 4x^7l\right) - 48x^3l + 24x^4 + 24x^2l^2$$

So, residual $R = \int_0^l L(x)\phi(x)dx = -\frac{1}{630}\beta^4 l^9 + \frac{4}{5}l^5 = 0$

$$\beta^4 l^4 = 504$$

$$\beta l = 4.738$$

Hence $\omega_1 = \left(\beta l\right)^2 \sqrt{\frac{EI}{ml^4}} = 22.45\sqrt{\frac{EI}{ml^4}}$

Now taking two admissible functions as $\varphi_1(x) = 1 - \cos\frac{2\pi x}{L}$ and $\varphi_2(x) = 1 - \cos\frac{4\pi x}{L}$, writing

$$\varphi = c_1\varphi_1(x) + c_2\varphi_2(x) = c_1\left(1 - \cos\frac{2\pi x}{L}\right) + c_2\left(1 - \cos\frac{4\pi x}{L}\right)$$

$$R_i = \int_0^l L(x)\varphi_i(x)dx = 0$$

$$L(x) = c_1\left((r^2 - \beta^2)\cos rx + \beta^4\right) + c_2\left((16r^2 - \beta^2)\cos 2rx + \beta^4\right)$$

where $\beta^4 = m\omega^2 / EI$ and $r = 2\pi / L$

$$R_1 = \int_0^l \left(c_1\left((r^2 - \beta^2)\cos rx + \beta^4\right) + c_2\left((16r^2 - \beta^2)\cos 2rx + \beta^4\right)\right)(1 - \cos\frac{2\pi x}{L})dx$$

$$R_2 = \int_0^l \left(c_1\left((r^2 - \beta^2)\cos rx + \beta^4\right) + c_2\left((16r^2 - \beta^2)\cos 2rx + \beta^4\right)\right)\left(1 - \cos\frac{4\pi x}{L}\right)dx$$

By integration

$$R_1 = \left(\frac{r^4}{2} - 1.5\beta^4\right)c_1 - \beta^4 c_2$$

$$R_2 = \beta^4 c_1 + (1.5\beta^4 - 0.5r^4)c_2 = 0$$

$$\begin{bmatrix} 8\frac{\pi^4}{L^4} - \frac{3}{2}\beta^4 & -\beta^4 \\ \beta^4 & \frac{3}{2}\beta^4 - 128\frac{\pi^4}{L^4} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \Rightarrow [A]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

For frequency, Determinant of [A] =0

Taking $\beta^4 L^4 = \lambda$, the above determinant can be written as

$$5r^2 - 816\pi^4 r + 4096\pi^8 = 0$$

Hence frequency equation are for $\beta L = 4.741$ and $\beta L = 11.140$ for both ends fixed beam

$$\omega_1 = 22.48\left[\frac{EI}{ml^4}\right]^{1/2}, \quad c_1 = 23, c_2 = 1,$$

and $\omega_2 = 124.1\left[\frac{EI}{ml^4}\right]^{1/2}, c_1 = -0.69, \quad c_2 = 1$.

The two modes are

$$\varphi_1(x) = 23(1-\cos\frac{2\pi x}{l}) + \left(1-\cos\frac{4\pi x}{l}\right)$$

$$\varphi_2(x) = -0.69(1-\cos\frac{2\pi x}{l}) + \left(1-\cos\frac{4\pi x}{l}\right)$$

Matrix iteration method

This method is used to determine the natural frequencies and mode shapes of a multi degree of freedom system. As it is known that for a multi degree of freedom system, the governing equation can be reduced to the eigenvalue problem given by

$$[A]\{X\} = \lambda\{X\} \qquad (1)$$

where $[A]=[M]^{-1}[K]$ is known as the dynamic matrix, λ is the eigen value and $\{X\}$ is the mode shape. From this equation it may be noted that any normal mode when multiplied with the dynamic matrix will reproduce itself. In matrix iteration method, assumed displacement of the masses are used to get the calculated displacement. This is repeated till equation (1) is satisfied.

Steps used in the matrix iteration method.

- Assume a value of the modal vector. (for example 3:2:1 for 3 dof system)
- Substitute the assumed value in left hand side of equation (1) and simplify to obtain a ratio (for example the obtained value is 4:3:1).
- If the value obtained in step II is same as the assumed value, then it is accepted as the correct modal value. Otherwise, the obtained value is substituted as the trial value and the second step is repeated till the correct modal value is obtained.
- After getting the modal values, from equation (1) the corresponding eigenvalue can be obtained.

In general matrix iteration method would converge to the fundamental mode. If the assumed system of displacements does not include the fundamental mode then the matrix iteration will converge to the next higher mode contained in the assumed system of displacements. Orthogonality principle is used to sweep out the unwanted modes from assumed displacements.

In case of semi-definite systems, rigid body mode (zero frequency) is also present. For such cases constraint matrices can be constructed to sweep out rigid body component of the absolute motion.

Estimation of higher mode frequencies

When the equations of motion are formulated in terms of the flexibility influence coefficients, the iteration procedure converges to the lowest mode present in the assumed deflection. Hence in the absence of the lowest mode of the assumed deflection the iteration process will converge to the next lowest, or the second mode. Let the displacement X be written as the combination of eigenfunctions X_i as follows.

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n \quad \text{--(a)}$$

For example for a 3dof system $X = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}'$ and $X_i = \{x_1, x_2, x_3\}_i'$

To remove the first mode, one has to impose the condition $c_1 = 0$. To do this, premultiplying $X_1'M$ in both sides of equation (a) and applying orthogonality principles of normal modes yields

$$X_1'MX = c_1 X_1'MX_1 + c_2 X_1'MX_2 + \cdots + c_n X_1'MX_n = c_1 X_1'MX_1 \qquad \text{(b)}$$

Hence for c_1 to be zero $X_1'MX = 0$ (c)

Using this condition a sweeping matrix S can be generated to sweep out the lowest mode from the assume displacement and the resulting iteration will lead to the higher mode.

In case of 3 dof system, expression (c)leads to

$$X_1'MX = \{x_1, x_2, x_3\} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} = 0$$

$$m_1 x_1 \bar{x}_1 + m_2 x_2 \bar{x}_2 + m_3 x_3 \bar{x}_3 = 0.$$

So one may write the above equation as

$$\bar{x}_1 = -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right)\bar{x}_2 - \frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right)\bar{x}_3$$
$$\bar{x}_2 = \bar{x}_2$$
$$\bar{x}_3 = \bar{x}_3$$

or in matrix form

$$\begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right) & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix}$$

or, $X = SX$

where $S = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right) & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

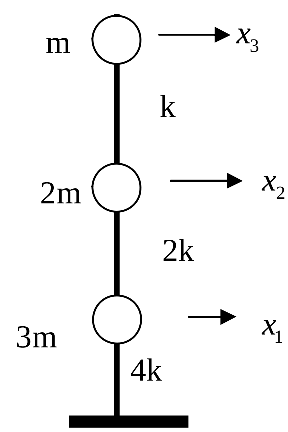
S is known as the sweeping matrix, which will eliminate the presence of the fundamental mode. Similarly one should substitute $c_1 = c_2 = 0$ to eliminate the first two normal modes from the assumed vibration to determine the 3rd mode frequency.

Example

Consider a long beam with three masses as shown in figure 1. Determine the mode shapes and natural frequencies of the system using matrix iteration method.

Solution

To Determine natural frequency and mode shape of multi degree of freedom system, the influence coefficients are obtained as



$$a_{11} = a_{21} = a_{31} = \frac{1}{4k}$$

$$a_{11} = a_{21} = a_{31} = a_{12} = a_{13} = \frac{1}{4k}$$

$$a_{22} = \frac{1}{4k} + \frac{1}{2k} = \frac{3}{4k}$$

$$a_{22} = a_{23} = a_{32} = \frac{3}{4k}$$

$$a_{33} = \frac{1}{4k} + \frac{1}{2k} + \frac{1}{k} = \frac{7}{4k}$$

The flexibility influence coefficient matrix can be written as

$$a = \frac{1}{4k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 3 & 7 \end{bmatrix}.$$

The mass matrix can be written as $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence the displacements at different position due to inertia forces are

$$-x_1 = a_{11} 3m\ddot{x}_1 + a_{12} 2m\ddot{x}_2 + a_{13} m\ddot{x}_3$$

$$-x_2 = a_{21}3m\ddot{x}_1 + a_{22}2m\ddot{x}_2 + a_{23}m\ddot{x}_3$$
$$-x_3 = a_{31}3m\ddot{x}_1 + a_{32}2m\ddot{x}_2 + a_{33}m\ddot{x}_3$$

Now substituting $\ddot{x}_i = -w^2 x_i$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Now taking

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Now taking any a trial vector $X = \{1 \quad 2 \quad 3\}'$

$$\begin{Bmatrix} 1 \\ 2.4893 \\ 3.7735 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix}\begin{Bmatrix} 1 \\ 2.4893 \\ 3.7735 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{Bmatrix} 11.7521 \\ 29.2565 \\ 44.3503 \end{Bmatrix}$$

$$\begin{Bmatrix} 1 \\ 2.4893 \\ 3.7735 \end{Bmatrix} = \frac{11.7521m\omega^2}{4k}\begin{Bmatrix} 1 \\ 2.4895 \\ 3.7738 \end{Bmatrix}$$

Hence, it may be noted that the assumed mode shape matches with the obtained mode shape upto 3rd decimal. So one can take the first normal mode X_1 as

$$X_1 = \begin{Bmatrix} 1 \\ 2.4895 \\ 3.7738 \end{Bmatrix}$$

and $\frac{11.7521m\omega^2}{4k} = 1$

or, $\lambda_1 = \omega_1^2 = 0.3404\frac{k}{m}$

To find the second mode, using the sweeping matrix

$$S = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right) & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1.659 & -1.2579 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the new equation for second mode iteration is

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix}\begin{bmatrix} 0 & -1.659 & -1.2579 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

or, $$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k}\begin{bmatrix} 0 & -2.98 & -2.7740 \\ 0 & 1.02 & -0.7740 \\ 0 & 1.02 & 3.226 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Starting with a trial value of $X = \{1 \quad 0 \quad 1\}'$ after several iteration

$$\begin{Bmatrix} 1.0 \\ 0.8367 \\ -1.8996 \end{Bmatrix} = \frac{2.7756m\omega^2}{4k}\begin{Bmatrix} 1.0 \\ 0.8361 \\ -1.8988 \end{Bmatrix}$$

As the left hand side vector and right hand side vector are matching upto 3rd decimal we can take the 2nd normal mode as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.8361 \\ -1.8988 \end{Bmatrix}$$

$$\frac{2.7756m\omega^2}{4k} = 1$$

Hence the 2nd eigenvalues is $\lambda_2 = \omega_2^2 = \frac{4}{2.7756}\frac{k}{m} = 1.4411\frac{k}{m}$

Or, the second mode frequency $= \omega_2 = 1.2\sqrt{\frac{k}{m}}$

For the determination of third mode, one may impose the condition $c_1 = c_2 = 0$.

$$c_1 = \sum_{i=1}^{3} m_i (x_i)_1 \bar{x}_i = 0$$
$$c_2 = \sum_{i=1}^{3} m_i (x_i)_2 \bar{x}_i = 0$$
$$c_1 = m_1 (x_1)_1 \bar{x}_1 + m_2 (x_2)_1 \bar{x}_2 + m_3 (x_3)_1 \bar{x}_3$$
$$c_2 = m_1 (x_1)_2 \bar{x}_1 + m_2 (x_2)_2 \bar{x}_2 + m_3 (x_3)_2 \bar{x}_3$$

or
$$c_1 = 3\bar{x}_1 + 4.979\bar{x}_2 + 3.7738\bar{x}_3$$
$$c_2 = 3\bar{x}_1 + 1.6722\bar{x}_2 - 1.8998\bar{x}_3$$

$$\bar{x}_1 = 1.5896\,\bar{x}_3$$
$$\bar{x}_2 = -1.7157\,\bar{x}_3$$
$$\begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1.5896 \\ 0 & 0 & -1.7157 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix}$$

Hence one may use the following sweeping matrix to eliminate the first two modes

$$S_2 = \begin{bmatrix} 0 & 0 & 1.5896 \\ 0 & 0 & -1.7157 \\ 0 & 0 & 1 \end{bmatrix}$$

So for third mode the matrix iteration equation will be

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k} \begin{bmatrix} 3 & 2 & 1 \\ 3 & 6 & 3 \\ 3 & 6 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1.5896 \\ 0 & 0 & -1.7157 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

or, $$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{4k} \begin{bmatrix} 0 & 0 & 2.3374 \\ 0 & 0 & -2.5254 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

It can immediately be observed that the third mode converges to the last column of the sweeping matrix.

$$\begin{Bmatrix} 1.5896 \\ -1.7157 \\ 1 \end{Bmatrix} = 1.4746 \frac{m\omega^2}{4k} \begin{Bmatrix} 1.5896 \\ -1.7157 \\ 1 \end{Bmatrix}$$

$$X_3 = \begin{Bmatrix} 1.5896 \\ -1.7157 \\ 1 \end{Bmatrix}$$

$$1.4746 \frac{m\omega^2}{4k} \begin{Bmatrix} 1.5896 \\ -1.7157 \\ 1 \end{Bmatrix} = 1$$

$$\lambda_3 = \omega_3^2 = \frac{4}{1.4742} \frac{k}{m} = 2.7133 \frac{k}{m}$$

So, $\omega_3 = 1.6472\sqrt{\frac{k}{m}}$.

So the normal mode frequencies are

$$\omega_1 = 0.5834\sqrt{\frac{k}{m}} \qquad \omega_2 = 1.2\sqrt{\frac{k}{m}} \qquad \omega_3 = 1.6472\sqrt{\frac{k}{m}}$$

$$X_1 = \begin{Bmatrix} 1 \\ 2.4895 \\ 3.7738 \end{Bmatrix} \quad X_2 = \begin{Bmatrix} 1.0 \\ 0.8361 \\ -1.8988 \end{Bmatrix} \qquad X_3 = \begin{Bmatrix} 1.5896 \\ -1.7157 \\ 1 \end{Bmatrix}$$